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LARGE DEFORMATIONS, SUPERPOSED SMALL DEFORMATIONS AND STABILITY OF ELASTIC RODS

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Abstract—We give a theory for the small deformations superposed on the large deformation of an elastic rod. We consider some aspects of material and geometrical symmetry and discuss the solution of some problems of finite deformation. Also we discuss the stability of a straight rod which has been subjected to a large simple extension.

1. INTRODUCTION

STARTING with the general thermodynamical theory of elastic rods given by Green and Laws [1], we develop a theory of small deformations superposed on a large elastic deformation of the rod. We go on to discuss certain symmetries associated with an elastic rod and obtain the reduced form of the constitutive equations in this case. It turns out that the occurrence of these symmetries implies that one can obtain the general solutions of some problems of finite deformation—this is analogous to the situation in the full three dimensional theory of elasticity. We discuss the solution to the problem of the extension and torsion of an initially straight rod and to the problem of the flexure of an initially straight rod.

Next we give the complete set of equations governing the small displacement of a straight rod subjected to a (large) simple extension. Here, we find that the equations separate into four distinct groups, two concerned with flexure, one with torsion and one with longitudinal extension. Also the temperature occurs only in the last of these groups. This parallels the result of Green *et al.* [2] in the linear theory of straight elastic rods.

We conclude the paper with a stability discussion. We consider the stability of a straight rod which has undergone a simple extension when the rod is simply supported or when the rod is clamped at both ends. The critical values obtained for the compressive force in the rod are, under some apparently reasonable assumptions, less than the values obtained using the classical theory.

2. AN ELASTIC ROD

A rod is defined by Green and Laws [1] to be a curve c, embedded in Euclidean 3-space, at each point of which there are two assigned directors. Let c be defined by

$$\mathbf{r} = \mathbf{r}(\theta, t), \tag{2.1}$$

where $\mathbf{r}$ is the position vector, relative to a fixed origin, of a point on c and t denotes the time. We regard θ as a convected coordinate defining points on the curve. The initial position of c is denoted by $\overline{\mathscr{C}}$. Also, let two directors $\bar{\mathbf{A}}_\alpha$ ($\alpha = 1, 2$) be assigned to every point of $\overline{\mathscr{C}}$. The duals of $\bar{\mathbf{A}}_\alpha$ at time t are denoted by $\mathbf{a}_\alpha$ and the motion of the rod is given by

$$\mathbf{r} = \mathbf{r}(\theta, t), \qquad \mathbf{a}_\alpha = \mathbf{a}_\alpha(\theta, t). \tag{2.2}$$

We define $\mathbf{a}_3$, $\bar{\mathbf{A}}_3$ through

$$\mathbf{a}_3 = \mathbf{a}_3(\theta, t) = \partial\mathbf{r}/\partial\theta, \qquad \bar{\mathbf{A}}_3 = \mathbf{a}_3(\theta, 0), \tag{2.3}$$

and assume that

$$[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] > 0. \tag{2.4}$$

In the subsequent work we use a usual index notation in which Latin indices have the values 1, 2, 3, Greek indices the values 1, 2 and repeated indices are summed over the appropriate range. It is, perhaps, worth remarking that Greek indices are not always "tensor" indices.

In view of (2.4) we may define a set of reciprocal base vectors $\mathbf{a}^i$ by

$$\mathbf{a}^i \,.\, \mathbf{a}_j = \delta^i_j, \tag{2.5}$$

where δ^i_j is the Kronecker delta, and use the notation

$$a_{ij} = \mathbf{a}_i \,.\, \mathbf{a}_j, \qquad a^{ij} = \mathbf{a}^i \,.\, \mathbf{a}^j, \qquad a = |a_{ij}|, \tag{2.6}$$

$$\frac{\partial \mathbf{a}_i}{\partial\theta} = \kappa_{ir}\mathbf{a}^r = \kappa_i^{\cdot r}\mathbf{a}_r, \qquad \kappa_i^{\cdot r} = a^{rs}\kappa_{is}. \tag{2.7}$$

The local equation of mass conservation is

$$\rho\sqrt{a_{33}} = \text{function of } \theta = \beta(\theta), \text{ say}, \tag{2.8}$$

where ρ is the mass per unit length of c. The equations of motion are

$$\frac{\partial \mathbf{n}}{\partial\theta} + \beta\mathbf{f} = \beta\ddot{\mathbf{r}}, \tag{2.9}$$

$$\frac{\partial \mathbf{a}_\alpha}{\partial\theta} \times \mathbf{p}^\alpha + \mathbf{a}_3 \times \mathbf{n} + \mathbf{a}_\alpha \times \boldsymbol{\pi}^\alpha = \mathbf{0}, \tag{2.10}$$

where the two vectors $\boldsymbol{\pi}^\alpha$ are defined by†

$$\boldsymbol{\pi}^\alpha = \frac{\partial \mathbf{p}^\alpha}{\partial\theta} + \beta\mathbf{q}^\alpha, \qquad \mathbf{q}^\alpha = \mathbf{l}^\alpha - y^{\alpha\beta}\ddot{\mathbf{a}}_\beta. \tag{2.11}$$

In the preceding equations $\mathbf{n}$ is the force and $\mathbf{p}^\alpha$ the director forces which constitute the mechanical action in the rod. Also $\mathbf{f}$ is the assigned force‡ per unit mass, $\mathbf{q}^\alpha$ the difference between the assigned director force‡ and the director inertia terms and a superposed dot denotes the material derivative with respect to t holding θ fixed.

It is often more convenient to use the component form of the equations of motion. Thus, if

$$n^i = \mathbf{n} \,.\, \mathbf{a}^i, \qquad f^i = \mathbf{f} \,.\, \mathbf{a}^i, \qquad \pi^{\alpha i} = \boldsymbol{\pi}^\alpha \,.\, \mathbf{a}^i, \qquad p^{\alpha i} = \mathbf{p}^\alpha \,.\, \mathbf{a}^i, \qquad c^i = \ddot{\mathbf{r}} \,.\, \mathbf{a}^i, \tag{2.12}$$

† The vectors $\boldsymbol{\pi}^\alpha$ used here are *not* the same as those used by Green and Laws [1].
‡ That is load plus body force.

we see from (2.9) to (2.11) that

$$\frac{\partial n^i}{\partial \theta}+\kappa_r^{\cdot i}n^r+\beta f^i = \beta c^i, \tag{2.13}$$

and

$$\pi^{\alpha\beta}-\pi^{\beta\alpha}+p^{\gamma\beta}\kappa_\gamma^{\cdot\alpha}-p^{\gamma\alpha}\kappa_\gamma^{\cdot\beta} = 0, \tag{2.14}$$

$$\pi^{\beta 3}+p^{\alpha 3}\kappa_\alpha^{\cdot\beta}-p^{\alpha\beta}\kappa_\alpha^{\cdot 3}-n^\beta = 0. \tag{2.15}$$

For an elastic rod, the Helmholtz free energy per unit mass, A, is given by

$$A = A(T,\gamma_{ij},\sigma_{\alpha i},\bar{A}_{ij},\bar{K}_{\alpha i}), \tag{2.16}$$

where

$$\gamma_{ij} = a_{ij}-\bar{A}_{ij}, \qquad \sigma_{ij} = \kappa_{ij}-\bar{K}_{ij}, \tag{2.17}$$

$\bar{A}_{ij}$, $\bar{K}_{ij}$ denote the initial values of a_{ij}, κ_{ij} respectively, and T denotes the temperature. Also

$$n^3-p^{\alpha 3}\kappa_\alpha^{\cdot 3} = 2\beta\frac{\partial A}{\partial \gamma_{33}}, \tag{2.18}$$

$$n^\beta-p^{\alpha 3}\kappa_\alpha^{\cdot\beta} = \beta\frac{\partial A}{\partial \gamma_{\beta 3}}, \tag{2.19}$$

$$\pi^{\alpha\beta}+\pi^{\beta\alpha}-p^{\gamma\beta}\kappa_\gamma^{\cdot\alpha}-p^{\gamma\alpha}\kappa_\gamma^{\cdot\beta} = 4\beta\frac{\partial A}{\partial \gamma_{\alpha\beta}}, \tag{2.20}$$

$$p^{\alpha i} = \beta\frac{\partial A}{\partial \sigma_{\alpha i}}, \tag{2.21}$$

$$S = -\frac{\partial A}{\partial T}, \tag{2.22}$$

where S denotes the entropy per unit mass. In addition the residual energy equation is

$$\beta r-\beta T\dot{S}-\partial h/\partial\theta = 0, \tag{2.23}$$

where

$$-h\frac{\partial T}{\partial \theta} \geq 0, \tag{2.24}$$

r is the heat supply function per unit mass per unit time, and h is the flux of heat along c per unit time. In evaluating the right hand sides of (2.18) to (2.22), A is to be regarded as a function of $\gamma_{\beta 3}, \gamma_{33}, \frac{1}{2}(\gamma_{\alpha\beta}+\gamma_{\beta\alpha})$. Finally the constitutive equation for h is

$$h = h(T,\gamma_{ij},\sigma_{\alpha i},\bar{A}_{ij},\bar{K}_{\alpha i},\partial T/\partial\theta), \tag{2.25}$$

and provided h is a continuous function of $\partial T/\partial\theta$ in the neighbourhood of $\partial T/\partial\theta = 0$, we may show from (2.24) that

$$h = 0 \quad \text{whenever} \quad \partial T/\partial\theta = 0. \tag{2.26}$$

3. SMALL DEFORMATION SUPERPOSED ON A LARGE DEFORMATION

We consider three configurations of the rod: the initial configuration with the position of the curve $\bar{\mathscr{C}}$ and the directors denoted by $\bar{\mathbf{A}}_\alpha$, the first deformed configuration where the curve is denoted by $\mathscr{C}$ and the directors by $\mathbf{A}_\alpha$, the final configuration in which the curve is denoted by c and the directors by $\mathbf{a}_\alpha$. In the initial configuration we specify the curve $\bar{\mathscr{C}}$ by

$$\bar{\mathbf{R}} = \bar{\mathbf{R}}(\theta), \tag{3.1}$$

and assume that, in this configuration, the rod is in equilibrium at uniform temperature T_0 and entropy S_0. We also assume that the first deformed configuration is one of equilibrium at uniform temperature T_1 and entropy S_1, and that $\mathscr{C}$ is specified by

$$\mathbf{R} = \mathbf{R}(\theta). \tag{3.2}$$

The final configuration is obtained by a subsequent small deformation with the curve c given by

$$\mathbf{r} = \mathbf{r}(\theta, t) = \mathbf{R}(\theta) + \varepsilon\mathbf{u}(\theta, t), \tag{3.3}$$

and the directors $\mathbf{a}_\alpha$ are determined by

$$\mathbf{a}_\alpha = \mathbf{a}_\alpha(\theta, t) = \mathbf{A}_\alpha(\theta) + \varepsilon\mathbf{b}_\alpha(\theta, t). \tag{3.4}$$

where ε is a small real parameter. Hence the displacements and director displacements, from the first to the second deformed configuration are $\varepsilon\mathbf{u}$, $\varepsilon\mathbf{b}_\alpha$ respectively. In the following analysis, powers of ε above the first will be neglected—except in the free energy A.

From (2.3), (3.3) and (3.4) we observe that

$$\left.\begin{aligned} \mathbf{a}_i &= \mathbf{A}_i + \varepsilon\mathbf{b}_i, \\ \mathbf{b}_3 &= \partial\mathbf{u}/\partial\theta, \qquad \mathbf{A}_3 = \partial\mathbf{R}/\partial\theta. \end{aligned}\right\} \tag{3.5}$$

We shall denote the quantities occurring in (2.6) and (2.7) which refer to the initial undeformed configuration by majuscules with a superposed bar, for example $\bar{K}_{ij}$. Also those kinematic quantities which refer to the first deformed configuration will be denoted by majuscules, for example A_{ij}. Using this notation we have, from (2.5), (2.6), (2.7) and (2.17)

$$a_{ij} = A_{ij} + \varepsilon(b_{ij} + b_{ji}), \qquad \gamma_{ij} = \Gamma_{ij} + \varepsilon(b_{ij} + b_{ji}), \tag{3.6}$$

where

$$\Gamma_{ij} = A_{ij} - \bar{A}_{ij}, \tag{3.7}$$

and

$$\mathbf{b}_i = b_{ij}\mathbf{A}^j = b_i^{\cdot j}\mathbf{A}_j, \qquad b_i^{\cdot j} = A^{kj}b_{ik}. \tag{3.8}$$

Also

$$\mathbf{a}^i = \mathbf{A}^i - \varepsilon b_j^{\cdot i}\mathbf{A}^j, \tag{3.9}$$

and, recalling (2.17),

$$\kappa_{ij} = K_{ij} + \varepsilon\lambda_{ij}, \qquad \sigma_{ij} = \Sigma_{ij} + \varepsilon\lambda_{ij}, \tag{3.10}$$

with

$$\lambda_{ij} = A_{js}\frac{\partial b_i^{\cdot s}}{\partial \theta} + b_i^{\cdot s}K_{sj} + b_j^{\cdot s}K_{is}, \tag{3.11}$$

$$K_{ij} = \mathbf{A}_j \cdot \frac{\partial \mathbf{A}_i}{\partial \theta}, \qquad \Sigma_{ij} = K_{ij} - \bar{K}_{ij}. \tag{3.12}$$

In addition

$$\kappa_i^{\cdot j} = K_i^{\cdot j} + \varepsilon \mu_i^{\cdot j}, \tag{3.13}$$

where

$$\mu_i^{\cdot j} = \frac{\partial b_i^{\cdot j}}{\partial \theta} + b_i^{\cdot k}K_k^{\cdot j} - b_k^{\cdot j}K_i^{\cdot k}, \tag{3.14}$$

$$K_i^{\cdot j} = A^{jk}K_{ik}. \tag{3.15}$$

We suppose that the values of $\mathbf{n}$, $\mathbf{p}^\alpha$ and $\boldsymbol{\pi}^\alpha$ in the first deformed configuration are $\mathbf{N}$, $\mathbf{P}^\alpha$ and $\boldsymbol{\Pi}^\alpha$ and write

$$\mathbf{n} = \mathbf{N} + \varepsilon \mathbf{v}, \qquad \mathbf{p}^\alpha = \mathbf{P}^\alpha + \varepsilon \boldsymbol{\xi}^\alpha, \qquad \boldsymbol{\pi}^\alpha = \boldsymbol{\Pi}^\alpha + \varepsilon \boldsymbol{\varpi}^\alpha \tag{3.16}$$

Also the assigned forces $\mathbf{f}$ and director forces $\mathbf{l}^\alpha$ are assumed to be of the form

$$\mathbf{f} = \mathbf{F} + \varepsilon \mathbf{f}', \qquad \mathbf{l}^\alpha = \mathbf{L}^\alpha + \varepsilon \mathbf{l}'^\alpha, \tag{3.17}$$

so that if

$$\mathbf{q}^\alpha = \mathbf{Q}^\alpha + \varepsilon \mathbf{q}'^\alpha, \tag{3.18}$$

then

$$\mathbf{Q}^\alpha = \mathbf{L}^\alpha, \qquad \mathbf{q}'^\alpha = \mathbf{l}'^\alpha - y^{\alpha\beta}\ddot{\mathbf{b}}_\beta, \tag{3.19}$$

since the first deformed configuration is one of equilibrium.

From (2.9) to (2.11) and (3.16) to (3.19) we obtain the following equations of equilibrium of the first deformed configuration:

$$\frac{\partial \mathbf{N}}{\partial \theta} + \beta \mathbf{F} = \mathbf{0}, \tag{3.20}$$

$$\frac{\partial \mathbf{A}_\alpha}{\partial \theta} \times \mathbf{P}^\alpha + \mathbf{A}_3 \times \mathbf{N} + \mathbf{A}_\alpha \times \boldsymbol{\Pi}^\alpha = \mathbf{0}, \tag{3.21}$$

with

$$\boldsymbol{\Pi}^\alpha = \frac{\partial \mathbf{P}^\alpha}{\partial \theta} + \beta \mathbf{L}^\alpha. \tag{3.22}$$

We also obtain the equations of motion for the subsequent small deformation in the form

$$\frac{\partial \mathbf{v}}{\partial \theta} + \beta \mathbf{f}' = \beta \ddot{\mathbf{u}}, \tag{3.23}$$

$$\frac{\partial \mathbf{A}_\alpha}{\partial \theta} \times \boldsymbol{\xi}^\alpha + \mathbf{A}_3 \times \mathbf{v} + \mathbf{A}_\alpha \times \boldsymbol{\varpi}^\alpha + \frac{\partial \mathbf{b}_\alpha}{\partial \theta} \times \mathbf{P}^\alpha + \mathbf{b}_3 \times \mathbf{N} + \mathbf{b}_\alpha \times \boldsymbol{\Pi}^\alpha = \mathbf{0}, \tag{3.24}$$

with

$$\boldsymbol{\varpi}^{\alpha} = \frac{\partial \boldsymbol{\xi}^{\alpha}}{\partial \theta} + \beta \mathbf{q}'^{\alpha}. \tag{3.25}$$

Next, we wish to determine the component form of the equations of equilibrium (3.20) to (3.22) and the equations of motion (3.23) to (3.25). If

$$\left.\begin{array}{l} \Pi^{\alpha i} = \boldsymbol{\Pi}^{\alpha} . \mathbf{A}^{i}, \qquad \varpi^{\alpha i} = \boldsymbol{\varpi}^{\alpha} . \mathbf{A}^{i}, \qquad P^{\alpha i} = \mathbf{P}^{\alpha} . \mathbf{A}^{i}, \\ v^{i} = \mathbf{v} . \mathbf{A}^{i}, \qquad \xi^{\alpha i} = \boldsymbol{\xi}^{\alpha} . \mathbf{A}^{i}, \end{array}\right\} \tag{3.26}$$

then from (2.12), (3.9), (3.16) and (3.26) we find that

$$\pi^{\alpha i} = \Pi^{\alpha i} + \varepsilon(\varpi^{\alpha i} - b_{k}^{\cdot i}\Pi^{\alpha k}), \tag{3.27}$$

$$n^{i} = N^{i} + \varepsilon(v^{i} - b_{k}^{\cdot i}N^{k}), \tag{3.28}$$

$$p^{\alpha i} = P^{\alpha i} + \varepsilon(\xi^{\alpha i} - b_{k}^{\cdot i}P^{\alpha k}). \tag{3.29}$$

With the help of these results, we may show from (3.20) to (3.22), or (2.13) to (2.15), that

$$\frac{\partial N^{i}}{\partial \theta} + K_{r}^{\cdot i}N^{r} + \beta F^{i} = 0, \tag{3.30}$$

$$\Pi^{\alpha\beta} - \Pi^{\beta\alpha} + P^{\gamma\beta}K_{\gamma}^{\cdot\alpha} - P^{\gamma\alpha}K_{\gamma}^{\cdot\beta} = 0, \tag{3.31}$$

$$\Pi^{\beta 3} + P^{\alpha 3}K_{\alpha}^{\cdot\beta} - P^{\alpha\beta}K_{\alpha}^{\cdot 3} - N^{\beta} = 0, \tag{3.32}$$

where, from (2.11) and (3.19)

$$\Pi^{\alpha i} = \frac{\partial P^{\alpha i}}{\partial \theta} + K_{r}^{\cdot i}P^{\alpha r} + \beta L^{\alpha i}, \tag{3.33}$$

$$L^{\alpha i} = \mathbf{L}^{\alpha} . \mathbf{A}^{i}. \tag{3.34}$$

Also the equations of motion (3.23) to (3.25), or (2.13) to (2.15), yield

$$\frac{\partial v^{i}}{\partial \theta} + K_{r}^{\cdot i}v^{r} + \beta f'^{i} = \beta \frac{\partial^{2} u^{i}}{\partial t^{2}}, \tag{3.35}$$

$$\begin{aligned} &\varpi^{\alpha\beta} - \varpi^{\beta\alpha} - b_{r}^{\cdot\beta}\Pi^{\alpha r} + b_{r}^{\cdot\alpha}\Pi^{\beta r} + P^{\gamma\beta}\mu_{\gamma}^{\cdot\alpha} - P^{\gamma\alpha}\mu_{\gamma}^{\cdot\beta} \\ &\quad + K_{\gamma}^{\cdot\alpha}(\xi^{\gamma\beta} - b_{r}^{\cdot\beta}P^{\gamma r}) - K_{\gamma}^{\cdot\beta}(\xi^{\gamma\alpha} - b_{r}^{\cdot\alpha}P^{\gamma r}) = 0, \end{aligned} \tag{3.36}$$

$$\begin{aligned} &\varpi^{\beta 3} - b_{r}^{\cdot 3}\Pi^{\beta r} - v^{\beta} + b_{r}^{\cdot\beta}N^{r} + K_{\alpha}^{\cdot\beta}(\xi^{\alpha 3} - b_{r}^{\cdot 3}P^{\alpha r}) \\ &\quad + \mu_{\alpha}^{\cdot\beta}P^{\alpha 3} - K_{\alpha}^{\cdot 3}(\xi^{\alpha\beta} - b_{r}^{\cdot\beta}P^{\alpha r}) - \mu_{\alpha}^{\cdot 3}P^{\alpha\beta} = 0, \end{aligned} \tag{3.37}$$

where

$$u^{i} = \mathbf{u} . \mathbf{A}^{i}, \tag{3.38}$$

$$\left.\begin{aligned} \varpi^{\alpha i} &= \frac{\partial \xi^{\alpha i}}{\partial \theta} + K_{r}^{\cdot i}\xi^{\alpha r} + \beta q'^{\alpha i}, \\ q'^{\alpha i} &= l'^{\alpha i} - y^{\alpha\beta}\frac{\partial^{2} b_{\beta}^{\cdot i}}{\partial t^{2}}. \end{aligned}\right\} \tag{3.39}$$

Finally we put

$$T = T_1+\varepsilon T', \qquad S = S_1+\varepsilon S', \qquad r = R+\varepsilon r', \qquad h = H+\varepsilon h'. \tag{3.40}$$

With the help of the results of this section we can now deduce the constitutive equations for the two deformed configurations of the rod. For the first deformed configuration, we find from (2.18) to (2.22) that

$$N^3 - P^{\alpha 3}K_{\alpha}^{\cdot 3} = 2\beta\frac{\partial A}{\partial \Gamma_{33}}, \tag{3.41}$$

$$N^{\beta} - P^{\alpha 3}K_{\alpha}^{\cdot \beta} = \beta\frac{\partial A}{\partial \Gamma_{\beta 3}}, \tag{3.42}$$

$$\Pi^{\alpha\beta}+\Pi^{\beta\alpha}-P^{\gamma\beta}K_{\gamma}^{\cdot\alpha}-P^{\gamma\alpha}K_{\gamma}^{\cdot\beta} = 4\beta\frac{\partial A}{\partial \Gamma_{\alpha\beta}}, \tag{3.43}$$

$$P^{\alpha i} = \beta\frac{\partial A}{\partial \Sigma_{\alpha i}}, \tag{3.44}$$

$$S_1 = -\frac{\partial A}{\partial T_1}, \tag{3.45}$$

where A is to be evaluated at this configuration. Also the energy equation reduces to

$$\beta R - \partial H/\partial\theta = 0. \tag{3.46}$$

But, since this first deformed configuration is one of uniform temperature T_1, we see from (2.26) that $H = 0$, and hence

$$\partial H/\partial\theta = 0. \tag{3.47}$$

Also (3.46) shows that for this configuration to be possible we need

$$R = 0. \tag{3.48}$$

The constitutive equations for the infinitesimal increments are found to be

$$\begin{aligned} &\nu^3 - b_r^{\cdot 3}N^r - K_{\alpha}^{\cdot 3}(\xi^{\alpha 3} - b_r^{\cdot 3}P^{\alpha r}) - P^{\alpha 3}\mu_{\alpha}^{\cdot 3} \\ &= 4\beta\frac{\partial^2 A}{\partial \Gamma_{33}^2}b_{33} + 2\beta\frac{\partial^2 A}{\partial \Gamma_{\beta 3}\partial \Gamma_{33}}(b_{\beta 3}+b_{3\beta}) + 2\beta\frac{\partial^2 A}{\partial \Gamma_{\alpha\beta}\partial \Gamma_{33}}(b_{\alpha\beta}+b_{\beta\alpha}) \\ &\quad + 2\beta\frac{\partial^2 A}{\partial \Sigma_{\alpha i}\partial \Gamma_{33}}\lambda_{\alpha i} + 2\beta\frac{\partial^2 A}{\partial T_1\partial \Gamma_{33}}T', \end{aligned} \tag{3.49}$$

$$\begin{aligned} &\nu^{\beta} - b_r^{\cdot \beta}N^r - K_{\alpha}^{\cdot \beta}(\xi^{\alpha 3} - b_r^{\cdot 3}P^{\alpha r}) - P^{\alpha 3}\mu_{\alpha}^{\cdot \beta} \\ &= 2\beta\frac{\partial^2 A}{\partial \Gamma_{33}\partial \Gamma_{\beta 3}}b_{33} + \beta\frac{\partial^2 A}{\partial \Gamma_{\alpha 3}\partial \Gamma_{\beta 3}}(b_{\alpha 3}+b_{3\alpha}) + \beta\frac{\partial^2 A}{\partial \Gamma_{\lambda\mu}\partial \Gamma_{\beta 3}}(b_{\lambda\mu}+b_{\mu\lambda}) \\ &\quad + \beta\frac{\partial^2 A}{\partial \Sigma_{\alpha i}\partial \Gamma_{\beta 3}}\lambda_{\alpha i} + \beta\frac{\partial^2 A}{\partial T_1\partial \Gamma_{\beta 3}}T', \end{aligned} \tag{3.50}$$

$$
\begin{aligned}
&\varpi^{\alpha\beta}+\varpi^{\beta\alpha}-b_r^{\cdot\beta}\Pi^{\alpha r}-b_r^{\cdot\alpha}\Pi^{\beta r}-P^{\gamma\beta}\mu_\gamma^{\cdot\alpha}-P^{\gamma\alpha}\mu_\gamma^{\cdot\beta}\\
&\quad -K_\gamma^{\cdot\alpha}(\xi^{\gamma\beta}-b_r^{\cdot\beta}P^{\gamma r})-K_\gamma^{\cdot\beta}(\xi^{\gamma\alpha}-b_r^{\cdot\alpha}P^{\gamma r})\\
&\quad = 8\beta\frac{\partial^2 A}{\partial\Gamma_{33}\partial\Gamma_{\alpha\beta}}b_{33}+4\beta\frac{\partial^2 A}{\partial\Gamma_{\lambda 3}\partial\Gamma_{\alpha\beta}}(b_{\lambda 3}+b_{3\lambda})+4\beta\frac{\partial^2 A}{\partial\Gamma_{\lambda\mu}\partial\Gamma_{\alpha\beta}}(b_{\lambda\mu}+b_{\mu\lambda})\\
&\quad +4\beta\frac{\partial^2 A}{\partial\Sigma_{\gamma i}\partial\Gamma_{\alpha\beta}}\lambda_{\gamma i}+4\beta\frac{\partial^2 A}{\partial T_1\partial\Gamma_{\alpha\beta}}T',
\end{aligned}
\tag{3.51}
$$

$$
\begin{aligned}
\xi^{\alpha i}-b_r^{\cdot i}p^{\alpha r} &= 2\beta\frac{\partial^2 A}{\partial\Gamma_{33}\partial\Sigma_{\alpha i}}b_{33}+\beta\frac{\partial^2 A}{\partial\Gamma_{\beta 3}\partial\Sigma_{\alpha i}}(b_{\beta 3}+b_{3\beta})\\
&\quad +\beta\frac{\partial^2 A}{\partial\Gamma_{\lambda\mu}\partial\Sigma_{\alpha i}}(b_{\lambda\mu}+b_{\mu\lambda})+\beta\frac{\partial^2 A}{\partial\Sigma_{\beta r}\partial\Sigma_{\alpha i}}\lambda_{\beta r}+\beta\frac{\partial^2 A}{\partial T_1\partial\Sigma_{\alpha i}}T',
\end{aligned}
\tag{3.52}
$$

$$
\begin{aligned}
S' &= -2\frac{\partial^2 A}{\partial\Gamma_{33}\partial T_1}b_{33}-\frac{\partial^2 A}{\partial\Gamma_{\beta 3}\partial T_1}(b_{\beta 3}+b_{3\beta})\\
&\quad -\frac{\partial^2 A}{\partial\Gamma_{\alpha\beta}\partial T_1}(b_{\alpha\beta}+b_{\beta\alpha})-\frac{\partial^2 A}{\partial\Sigma_{\alpha i}\partial T_1}\lambda_{\alpha i}-\frac{\partial^2 A}{\partial T_1^2}T',
\end{aligned}
\tag{3.53}
$$

where A is to be evaluated at the first deformed configuration. Also the energy equation is

$$\beta r'-\beta T_1\dot{S}'-\partial h'/\partial\theta = 0, \tag{3.54}$$

where, in view of (2.24),

$$h' = \eta\partial T/\partial\theta, \qquad \eta \le 0, \tag{3.55}$$

and η depends upon the first deformed configuration. We do not write down the rather long expression for the free energy of the final deformed configuration of the rod.

This completes the general theory. We note that the preceding theory includes, rather trivially, the case of infinitesimal deformations of an elastic rod which is initially curved, force-free and at uniform temperature and entropy. The resulting equations, for the case of an initially straight rod, reduce to those of Green, Laws and Naghdi [2].

4. SYMMETRIES

It is well known that in the three dimensional theory of elasticity, there are relatively few solutions of the equilibrium equations for arbitrary free energy. However, if one introduces some symmetry restrictions upon the possible forms of the free energy then much more progress can be made. The same situation holds in the theory of elastic rods.

We consider, in the general theory, the free energy given by (2.16) but assume that A does not depend upon $\bar{A}_{ij}$ and $\bar{K}_{\alpha i}$. Thus

$$A = A(T, \gamma_{ij}, \sigma_{\alpha i}). \tag{4.1}$$

We assume that the Helmholtz function (4.1) is invariant under the transformations

$$\theta \to \pm\theta, \qquad \mathbf{a}_1 \to \pm\mathbf{a}_1, \qquad \mathbf{a}_2 \to \pm\mathbf{a}_2, \tag{4.2}$$

where we may take any combination of + and −. The first transformation implies that

$$\mathbf{a}_3 \rightarrow \pm \mathbf{a}_3, \qquad \bar{\mathbf{A}}_3 \rightarrow \pm \bar{\mathbf{A}}_3, \tag{4.3}$$

and with the second and third transformations we must associate

$$\bar{\mathbf{A}}_1 \rightarrow \pm \bar{\mathbf{A}}_1, \qquad \bar{\mathbf{A}}_2 \rightarrow \pm \bar{\mathbf{A}}_2, \tag{4.4}$$

respectively. A straightforward, but rather tedious, calculation shows that if A is a polynomial then it must reduce to a polynomial in T and the following 45 invariants

$$\begin{aligned}
&\gamma_{11},\ \gamma_{22},\ \gamma_{33},\ \gamma_{12}^2,\ \gamma_{23}^2,\ \gamma_{13}^2,\ \gamma_{12}\gamma_{13}\gamma_{23},\\
&\sigma_{11}^2,\ \sigma_{11}\sigma_{22},\ \sigma_{22}^2,\ \sigma_{12}^2,\ \sigma_{12}\sigma_{21},\ \sigma_{21}^2,\ \sigma_{13}^2,\ \sigma_{23}^2,\\
&\sigma_{11}\sigma_{12}\sigma_{13}\sigma_{23},\ \sigma_{11}\sigma_{21}\sigma_{13}\sigma_{23},\ \sigma_{12}\sigma_{22}\sigma_{13}\sigma_{23},\ \sigma_{21}\sigma_{22}\sigma_{13}\sigma_{23},\\
&\gamma_{12}\sigma_{11}\sigma_{12},\ \gamma_{12}\sigma_{11}\sigma_{21},\ \gamma_{12}\sigma_{12}\sigma_{22},\ \gamma_{12}\sigma_{21}\sigma_{22},\ \gamma_{12}\sigma_{13}\sigma_{23},\\
&\gamma_{13}\sigma_{11}\sigma_{13},\ \gamma_{13}\sigma_{22}\sigma_{13},\ \gamma_{13}\sigma_{12}\sigma_{23},\ \gamma_{13}\sigma_{21}\sigma_{23},\\
&\gamma_{23}\sigma_{11}\sigma_{23},\ \gamma_{23}\sigma_{22}\sigma_{23},\ \gamma_{23}\sigma_{12}\sigma_{13},\ \gamma_{23}\sigma_{21}\sigma_{13},\\
&\gamma_{12}\gamma_{13}\sigma_{11}\sigma_{23},\ \gamma_{12}\gamma_{13}\sigma_{22}\sigma_{23},\ \gamma_{12}\gamma_{13}\sigma_{12}\sigma_{13},\ \gamma_{12}\gamma_{13}\sigma_{21}\sigma_{13},\\
&\gamma_{12}\gamma_{23}\sigma_{11}\sigma_{13},\ \gamma_{12}\gamma_{23}\sigma_{22}\sigma_{13},\ \gamma_{12}\gamma_{23}\sigma_{12}\sigma_{23},\ \gamma_{12}\gamma_{23}\sigma_{21}\sigma_{23},\\
&\gamma_{13}\gamma_{23}\sigma_{11}\sigma_{12},\ \gamma_{13}\gamma_{23}\sigma_{11}\sigma_{21},\ \gamma_{13}\gamma_{23}\sigma_{12}\sigma_{22},\ \gamma_{13}\gamma_{23}\sigma_{21}\sigma_{22},\ \gamma_{13}\gamma_{23}\sigma_{13}\sigma_{23}.
\end{aligned} \tag{4.5}$$

The kinetic energy of the rod is

$$\tfrac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \tfrac{1}{2} y^{\alpha\beta} \dot{\mathbf{a}}_\alpha \cdot \dot{\mathbf{a}}_\beta \tag{4.6}$$

per unit mass and we demand that this is also invariant under the static transformations (4.2). Hence

$$y^{12} = y^{21} = 0. \tag{4.7}$$

5. LARGE DEFORMATION OF AN INITIALLY STRAIGHT ROD

In the first problem considered here, the initial curve $\overline{\mathscr{C}}$ and the curve $\mathscr{C}$ are straight lines. We choose the directors $\bar{\mathbf{A}}_\alpha$, which are associated with $\overline{\mathscr{C}}$, and the convected coordinate θ so that $\bar{\mathbf{A}}_i$ are a set of orthonormal vectors which are independent of θ. Hence

$$\bar{\mathbf{R}} = \theta \bar{\mathbf{A}}_3, \qquad \bar{A}_{ij} = \bar{\mathbf{A}}_i \cdot \bar{\mathbf{A}}_j = \delta_{ij}, \qquad \bar{K}_{ij} = 0. \tag{5.1}$$

We assume that the first deformed configuration of the rod is given by

$$\left.\begin{aligned}
\mathbf{A}_1 &= \lambda_1 \bar{\mathbf{A}}_1 \cos\psi\theta + \lambda_1 \bar{\mathbf{A}}_2 \sin\psi\theta,\\
\mathbf{A}_2 &= -\lambda_2 \bar{\mathbf{A}}_1 \sin\psi\theta + \lambda_2 \bar{\mathbf{A}}_2 \cos\psi\theta,\\
\mathbf{R} &= \lambda_3 \bar{\mathbf{R}} = \lambda_3 \theta \bar{\mathbf{A}}_3,
\end{aligned}\right\} \tag{5.2}$$

where $\lambda_1, \lambda_2, \lambda_3$ and ψ are constants. The above deformation consists of (finite) extension and torsion. It follows that

$$\left.\begin{aligned} A_{11} = \lambda_1^2, \qquad A_{22} = \lambda_2^2, \qquad A_{33} = \lambda_3^2, \\ A_{ij} = 0, \qquad i \neq j, \end{aligned}\right\} \tag{5.3}$$

and hence

$$\left.\begin{aligned} \Gamma_{11} = \lambda_1^2-1, \qquad \Gamma_{22} = \lambda_2^2-1, \qquad \Gamma_{33} = \lambda_3^2-1, \\ \Gamma_{ij} = 0, \qquad i \neq j. \end{aligned}\right\} \tag{5.4}$$

Also

$$\begin{gathered} K_{11} = K_{22} = K_{33} = K_{13} = K_{23} = 0 \\ K_{12} = -K_{21} = \lambda_1\lambda_2\psi, \qquad K_{1}^{\cdot 2} = \frac{\lambda_1\psi}{\lambda_2}, \qquad K_{2}^{\cdot 1} = -\frac{\lambda_2\psi}{\lambda_1}, \end{gathered} \tag{5.5}$$

and the only non-zero Σ_{ij} are

$$\Sigma_{12} = -\Sigma_{21} = \lambda_1\lambda_2\psi. \tag{5.6}$$

If the density of the initial rod is $\bar{\rho}$, then from (2.8) we have

$$\bar{\rho} = \rho\lambda_3 = \beta. \tag{5.7}$$

We assume that the free energy of the rod is given by (4.5) and that the rod is initially homogeneous. Hence from (3.41) to (3.44) we have

$$N^3 = 2\beta\frac{\partial A}{\partial \Gamma_{33}}, \qquad N^\beta = 0, \tag{5.8}$$

$$P^{11} = P^{22} = P^{13} = P^{23} = 0, \tag{5.9}$$

$$P^{12} = \beta\frac{\partial A}{\partial \Sigma_{12}}, \qquad P^{21} = \beta\frac{\partial A}{\partial \Sigma_{21}}, \tag{5.10}$$

$$\Pi^{11} = P^{21}K_{2}^{\cdot 1} + 2\beta\frac{\partial A}{\partial \Gamma_{11}}, \tag{5.11}$$

$$\Pi^{22} = P^{12}K_{1}^{\cdot 2} + 2\beta\frac{\partial A}{\partial \Gamma_{22}}, \tag{5.12}$$

$$\Pi^{12} + \Pi^{21} = 0, \tag{5.13}$$

and N^3, P^{12}, P^{21}, Π^{11}, Π^{22}, $(\Pi^{12}+\Pi^{21})$ are constant. The equations of equilibrium (3.30) are satisfied with zero body force and (3.31) and (3.32) yield

$$\Pi^{12} - \Pi^{21} = 0, \qquad \Pi^{\beta 3} = 0, \tag{5.14}$$

and therefore, with the help of (5.13),

$$\Pi^{12} = \Pi^{21} = 0. \tag{5.15}$$

In the absence of body forces, we deduce from (3.33), (5.11) and (5.12) that

$$\left.\begin{aligned} K_2^{\cdot 1}(P^{12}-P^{21}) &= 2\beta\frac{\partial A}{\partial \Gamma_{11}},\\ K_1^{\cdot 2}(P^{21}-P^{12}) &= 2\beta\frac{\partial A}{\partial \Gamma_{22}},\end{aligned}\right\} \tag{5.16}$$

or

$$\left.\begin{aligned} \frac{\lambda_2\psi}{\lambda_1}\left(\frac{\partial A}{\partial \Sigma_{21}}-\frac{\partial A}{\partial \Sigma_{12}}\right) &= 2\frac{\partial A}{\partial \Gamma_{11}},\\ \frac{\lambda_1\psi}{\lambda_2}\left(\frac{\partial A}{\partial \Sigma_{21}}-\frac{\partial A}{\partial \Sigma_{12}}\right) &= 2\frac{\partial A}{\partial \Gamma_{22}}.\end{aligned}\right\} \tag{5.17}$$

If we are given λ_3 and ψ then (5.17) provides two equations for the determination of λ_1 and λ_2. We remark that without prior knowledge of the function A, we are unable to decide whether (5.17) has solutions, a unique solution or no solutions.

In the special case when there is no twist

$$\psi = 0$$

and in addition to (5.8), (5.9), (5.14) and (5.15) we obtain

$$\Pi^{11} = \Pi^{22} = P^{12} = P^{21} = 0. \tag{5.18}$$

Also, in place of (5.17) we find that

$$\frac{\partial A}{\partial \Gamma_{11}} = \frac{\partial A}{\partial \Gamma_{22}} = 0, \tag{5.19}$$

which are two equations for the determination of λ_1 and λ_2 when λ_3 is given.

The second problem discussed here is concerned with the finite extension and flexure of an initially straight rod. We take the initial curve $\bar{\mathscr{C}}$ to be a straight line and the deformed curve $\mathscr{C}$ to be the arc of a circle of radius b. We again choose the directors $\bar{\mathbf{A}}_\alpha$ and the convected coordinate θ so that

$$\bar{\mathbf{R}} = \theta\bar{\mathbf{A}}_3, \qquad \bar{A}_{ij} = \delta_{ij}, \qquad \bar{K}_{ij} = 0. \tag{5.20}$$

In this problem, we assume that the first deformed configuration of the rod is given by

$$\left.\begin{aligned} \mathbf{R} &= b\bar{\mathbf{A}}_3\sin\phi - b\bar{\mathbf{A}}_1(1-\cos\phi) \qquad (\phi = \lambda_3\theta/b),\\ \mathbf{A}_1 &= \lambda_1\bar{\mathbf{A}}_1\cos\phi + \lambda_1\bar{\mathbf{A}}_3\sin\phi,\\ \mathbf{A}_2 &= \lambda_2\bar{\mathbf{A}}_2,\\ \mathbf{A}_3 &= \lambda_3\bar{\mathbf{A}}_3\cos\phi - \lambda_3\bar{\mathbf{A}}_1\sin\phi,\end{aligned}\right\} \tag{5.21}$$

where $\lambda_1, \lambda_2, \lambda_3$ and b are constants. The deformation specified by (5.21) consists of uniform extension (with extension ratios $\lambda_1, \lambda_2, \lambda_3$) together with pure flexure in a plane normal to $\bar{\mathbf{A}}_2$. It is a straightforward matter to verify that

$$\left.\begin{aligned} A_{11} = \lambda_1^2, \qquad A_{22} = \lambda_2^2, \qquad A_{33} = \lambda_3^2,\\ \Gamma_{11} = \lambda_1^2-1, \qquad \Gamma_{22} = \lambda_2^2-1, \qquad \Gamma_{33} = \lambda_3^2-1,\\ A_{ij} = \Gamma_{ij} = 0, \qquad i \neq j.\end{aligned}\right\} \tag{5.22}$$

Also the only non-zero components of K_{ij} and Σ_{ij} are

$$\Sigma_{13} = K_{13} = \lambda_1 \lambda_3^2/b = -K_{31} = -\Sigma_{31}. \tag{5.23}$$

If the initial density is $\bar{\rho}$, then we again recover (5.7). By assuming that the rod has the symmetries discussed in section 4 and that the rod is initially homogeneous, we obtain from (3.41) to (3.44) and (4.5) the following results:

$$P^{11} = P^{22} = P^{12} = P^{21} = P^{23} = 0, \tag{5.24}$$

$$N^{\beta} = 0, \tag{5.25}$$

$$P^{13} = \beta \frac{\partial A}{\partial \Sigma_{13}}, \qquad N^3 = P^{13} K_1^{\cdot 3} + 2\beta \frac{\partial A}{\partial \Gamma_{33}}, \tag{5.26}$$

$$\Pi^{11} = 2\beta \frac{\partial A}{\partial \Gamma_{11}}, \qquad \Pi^{22} = 2\beta \frac{\partial A}{\partial \Gamma_{22}}, \tag{5.27}$$

$$\Pi^{12} + \Pi^{21} = 0. \tag{5.28}$$

With the help of (5.23), (5.24) and (5.26) we find from (3.33) that in the absence of body forces

$$\Pi^{11} = -\frac{\lambda_3^2}{\lambda_1 b} P^{13}, \qquad \Pi^{\alpha i} = 0 \quad \text{otherwise}. \tag{5.29}$$

Also the equations of motion (3.30) to (3.32) indicate that we need

$$N^3 = 0 \quad \text{when } K_3^{\cdot 1} \neq 0. \tag{5.30}$$

Thus, from (5.26), (5.27), (5.29) and (5.30) we must have

$$\left.\begin{aligned} \frac{\lambda_1}{b} \frac{\partial A}{\partial \Sigma_{13}} + 2\frac{\partial A}{\partial \Gamma_{33}} &= 0, \\ 2\frac{\partial A}{\partial \Gamma_{11}} + \frac{\lambda_3^2}{\lambda_1 b} \frac{\partial A}{\partial \Sigma_{13}} &= 0, \\ \frac{\partial A}{\partial \Gamma_{22}} &= 0, \end{aligned}\right\} \tag{5.31}$$

in order that the deformation (5.21) be possible. Equations (5.31) provide three equations to determine the three quantities $\lambda_1, \lambda_2, \lambda_3$ in terms of b. We again note that since we do not know the form of the function A, we cannot make a definitive statement about solutions of (5.31).

We note that it is not possible to obtain the solution for simple extension directly from the preceding calculation. The reason is that we need to know that $K_3^{\cdot 1} \neq 0$ to obtain (5.30) and certainly $K_3^{\cdot 1} = 0$ in simple extension.

6. SMALL DEFORMATION SUPERPOSED ON SIMPLE EXTENSION

We use the results of the preceding section to discuss the small deformations of a rod under simple extension. For the first deformed configuration the relevant formulae are

$$\left.\begin{aligned} N^{\beta} = P^{\alpha i} = \Pi^{\alpha i} = 0, &\qquad N^3 \neq 0, \\ K_{ij} = 0, &\qquad \Sigma_{ij} = 0, \\ \Gamma_{11}, \Gamma_{22}, \Gamma_{33} &\quad \text{constant}, \\ A_{ij} = \Gamma_{ij} = 0, &\qquad i \neq j. \end{aligned}\right\} \tag{6.1}$$

In the first place we write down the explicit form of the constitutive equations for the superposed small deformation and in doing so it is convenient to introduce the notation

$$\begin{aligned}
&k_1 = \beta\frac{\partial^2 A}{\partial \Gamma_{11}^2}, && k_2 = \beta\frac{\partial^2 A}{\partial \Gamma_{22}^2}, && k_3 = \beta\frac{\partial^2 A}{\partial \Gamma_{33}^2}, \\
&k_4 = 4\beta\frac{\partial^2 A}{\partial \Gamma_{12}^2}, && k_5 = \beta\frac{\partial^2 A}{\partial \Gamma_{23}^2}, && k_6 = \beta\frac{\partial^2 A}{\partial \Gamma_{13}^2}, \\
&k_7 = 2\beta\frac{\partial^2 A}{\partial \Gamma_{11}\partial \Gamma_{22}}, && k_8 = 2\beta\frac{\partial^2 A}{\partial \Gamma_{11}\partial \Gamma_{33}}, && k_9 = 2\beta\frac{\partial^2 A}{\partial \Gamma_{22}\partial \Gamma_{33}}, \\
&k_{10} = \beta\frac{\partial^2 A}{\partial \Sigma_{11}^2}, && k_{11} = \beta\frac{\partial^2 A}{\partial \Sigma_{22}^2}, && k_{12} = \beta\frac{\partial^2 A}{\partial \Sigma_{12}^2}, \\
&k_{13} = \beta\frac{\partial^2 A}{\partial \Sigma_{21}^2}, && k_{14} = 2\beta\frac{\partial^2 A}{\partial \Sigma_{12}\partial \Sigma_{21}}, && k_{15} = \beta\frac{\partial^2 A}{\partial \Sigma_{23}^2}, \\
&k_{16} = \beta\frac{\partial^2 A}{\partial \Sigma_{13}^2}, && k_{17} = 2\beta\frac{\partial^2 A}{\partial \Sigma_{11}\partial \Sigma_{22}}, && k_{18} = \beta\frac{\partial^2 A}{\partial T_1\partial \Gamma_{11}}, \\
&k_{19} = \beta\frac{\partial^2 A}{\partial T_1\partial \Gamma_{22}}, && k_{20} = \beta\frac{\partial^2 A}{\partial T_1\partial \Gamma_{33}}, && k_{21} = \beta\frac{\partial^2 A}{\partial T_1^2},
\end{aligned} \tag{6.2}$$

where A is to be evaluated at the configuration (6.1). We observe that all the second derivatives of A, except those listed above, vanish at the configuration (6.1) when A is given by (4.5). With the help of (6.1) and (6.2), we obtain from (3.36), (3.37) and (3.49) to (3.53)

$$\varpi^{13} = v^1 - b_3^{\cdot 1} N^3 = k_6(b_{13}+b_{31}), \tag{6.3}$$

$$\varpi^{23} = v^2 - b_3^{\cdot 2} N^3 = k_5(b_{23}+b_{32}), \tag{6.4}$$

$$\left.\begin{aligned} v^3 - b_3^{\cdot 3} N^3 &= 2k_8 b_{11} + 2k_9 b_{22} + 4k_3 b_{33} + 2k_{20} T', \\ \varpi^{11} &= 4k_1 b_{11} + 2k_7 b_{22} + 2k_8 b_{33} + 2k_{18} T', \\ \varpi^{22} &= 2k_7 b_{11} + 4k_2 b_{22} + 2k_9 b_{33} + 2k_{19} T', \end{aligned}\right\} \tag{6.5}$$

$$\varpi^{12} = \varpi^{21} = k_4(b_{12}+b_{21}), \tag{6.6}$$

$$\left.\begin{aligned}\xi^{11} &= k_{10}\lambda_{11}+\tfrac{1}{2}k_{17}\lambda_{22},\\ \xi^{22} &= \tfrac{1}{2}k_{17}\lambda_{11}+k_{11}\lambda_{22},\end{aligned}\right\} \tag{6.7}$$

$$\left.\begin{aligned}\xi^{12} &= k_{12}\lambda_{12}+\tfrac{1}{2}k_{14}\lambda_{21},\\ \xi^{21} &= \tfrac{1}{2}k_{14}\lambda_{12}+k_{13}\lambda_{21},\end{aligned}\right\} \tag{6.8}$$

$$\xi^{13} = k_{16}\lambda_{13}, \tag{6.9}$$

$$\xi^{23} = k_{15}\lambda_{23}, \tag{6.10}$$

$$S' = -(k_{21}/\beta)T'-2(k_{18}/\beta)b_{11}-2(k_{19}/\beta)b_{22}-2(k_{20}/\beta)b_{33}. \tag{6.11}$$

If we put

$$y^{11} = \alpha_1, \qquad y^{22} = \alpha_2, \tag{6.12}$$

then we find from (3.39) and (4.7) that

$$\varpi^{\alpha i} = \frac{\partial\xi^{\alpha i}}{\partial\theta}+\beta q'^{\alpha i}, \tag{6.13}$$

$$q'^{\gamma i} = l'^{\gamma i}-\alpha_\gamma\frac{\partial^2 b_\gamma^{\cdot i}}{\partial t^2} \quad (\gamma \text{ not summed}). \tag{6.14}$$

Also the equations of motion (3.35) reduce to

$$\frac{\partial v^i}{\partial\theta}+\beta f'^i = \beta\frac{\partial^2 u^i}{\partial t^2}. \tag{6.15}$$

To help in the interpretation of these equations we recall from section 3 that

$$b_i^{\cdot j} = A^{jk}b_{ik}, \qquad b_{3i} = \frac{\partial u_i}{\partial\theta}, \tag{6.16}$$

$$\lambda_{ij} = A_{js}\frac{\partial b_i^{\cdot s}}{\partial\theta} = \frac{\partial b_{ij}}{\partial\theta}, \tag{6.17}$$

since the quantities A_{ij} are independent of θ.

Inspection of the equations for the small deformation shows that they separate into four distinct groups, two concerned with flexure, one with torsion and one with longitudinal extension. Also the temperature occurs only in the last of these groups. We collect the relevant groups of equations below.

We consider flexure of the rod in the plane normal to $\mathbf{A}_1$. The equations are

$$\frac{\partial v^2}{\partial\theta}+\beta f'^2 = \beta A^{22}\frac{\partial^2 u_2}{\partial t^2}, \tag{6.18}$$

$$\varpi^{23} = v^2-A^{22}b_{32}N^3 = \frac{\partial\xi^{23}}{\partial\theta}+\beta\left(l'^{23}-\alpha_2 A^{33}\frac{\partial^2 b_{23}}{\partial t^2}\right), \tag{6.19}$$

$$v^2-A^{22}N^3b_{32} = k_5(b_{23}+b_{32}), \tag{6.20}$$

$$\xi^{23} = k_{15}\frac{\partial b_{23}}{\partial\theta}, \qquad b_{32} = \frac{\partial u_2}{\partial\theta}. \tag{6.21}$$

A similar set of equations determine the flexure of the rod in a plane normal to $\mathbf{A}_2$. Upon elimination of v^2 and ξ^{23}, equations (6.18) yield the following pair of equations:

$$\left.\begin{aligned} (k_5+A^{22}N^3)\frac{\partial^2 u_2}{\partial\theta^2}+k_5\frac{\partial b_{23}}{\partial\theta}+\beta f'^2 &= \beta A^{22}\frac{\partial^2 u_2}{\partial t^2},\\ k_{15}\frac{\partial^2 b_{23}}{\partial\theta^2}-k_5 b_{23}+\beta l'^{23}-k_5\frac{\partial u_2}{\partial\theta} &= \beta\alpha_2 A^{33}\frac{\partial^2 b_{23}}{\partial t^2}. \end{aligned}\right\} \tag{6.22}$$

It should be emphasized that k_5, k_{15} and N^3 depend upon A_{11}, A_{22} and A_{33}. However, if we use the technique of Green [3] we may regard the initial configuration and the first deformed configuration to be coincident. Then the preceding work still yields a small deformation theory in the presence of initial forces. In such a theory $k_1 \ldots k_{21}$ are constants, but we do not have any means of calculating the initial forces in the rod.

It is, perhaps, of interest to note that the classical equation for the "Euler strut" may be obtained as a special case of this theory. First we recall that the theory of the Euler strut requires that the initial force in the rod be prescribed. We therefore use the idea mentioned in the preceding paragraph to obtain the relevant equation when the initial force is obtained "without deformation". In this case there is no loss of generality in taking

$$A_{11} = A_{22} = A_{33} = 1.$$

To recover the classical theory we let

$$(b_{23}+b_{32}) \to 0, \qquad k_5 \to \infty, \quad \alpha_2 = 0, \tag{6.23}$$

with $(v^2-A^{22}b_{32}N^3)$ not being determined by the constitutive equation (6.20). In the absence of body forces and director body forces, equations (6.18), (6.19), (6.20) and (6.23) yield the following equation for u_2:

$$k_{15}\frac{\partial^4 u_2}{\partial\theta^4}-N^3\frac{\partial^2 u_2}{\partial\theta^2}+\beta\frac{\partial^2 u_2}{\partial t^2} = 0. \tag{6.24}$$

This is the usual equation for the Euler strut.

Next, we consider the torsional motion of the rod. The equations are:

$$\left.\begin{aligned} \varpi^{12} &= \varpi^{21} = k_4(b_{12}+b_{21}),\\ \xi^{12} &= k_{12}\frac{\partial b_{12}}{\partial\theta}+\tfrac{1}{2}k_{14}\frac{\partial b_{21}}{\partial\theta},\\ \xi^{21} &= \tfrac{1}{2}k_{14}\frac{\partial b_{12}}{\partial\theta}+k_{13}\frac{\partial b_{21}}{\partial\theta},\\ \varpi^{12} &= \frac{\partial\xi^{12}}{\partial\theta}+\beta l'^{12}-\beta\alpha_1 A^{22}\frac{\partial^2 b_{12}}{\partial t^2},\\ \varpi^{21} &= \frac{\partial\xi^{21}}{\partial\theta}+\beta l'^{21}-\beta\alpha_2 A^{11}\frac{\partial^2 b_{21}}{\partial t^2}. \end{aligned}\right\} \tag{6.25}$$

These equations are of the same form as those obtained for rods without initial force by Green *et al.* [2], but here the coefficients $k_4, k_{12}, k_{13}, k_{14}$ depend upon A_{11}, A_{22}, A_{33}.

Finally we examine the equations for extensional motion of the rod:

$$
\begin{aligned}
\frac{\partial v^3}{\partial \theta}+\beta f'^3 &= \beta A^{33}\frac{\partial^2 u_3}{\partial t^2},\\
\varpi^{11} &= \frac{\partial \xi^{11}}{\partial \theta}+\beta l'^{11}-\beta\alpha_1 A^{11}\frac{\partial^2 b_{11}}{\partial t^2},\\
\varpi^{22} &= \frac{\partial \xi^{22}}{\partial \theta}+\beta l'^{22}-\beta\alpha_2 A^{22}\frac{\partial^2 b_{22}}{\partial t^2},\\
v^3-A^{33}b_{33}N^3 &= 2k_8 b_{11}+2k_9 b_{22}+4k_3 b_{33}+2k_{20}T',\\
\varpi^{11} &= 4k_1 b_{11}+2k_7 b_{22}+2k_8 b_{33}+2k_{18}T',\\
\varpi^{22} &= 2k_7 b_{11}+4k_2 b_{22}+2k_9 b_{33}+2k_{19}T',\\
\xi^{11} &= k_{10}\frac{\partial b_{11}}{\partial \theta}+\tfrac{1}{2}k_{17}\frac{\partial b_{22}}{\partial \theta},\\
\xi^{22} &= \tfrac{1}{2}k_{17}\frac{\partial b_{11}}{\partial \theta}+k_{11}\frac{\partial b_{22}}{\partial \theta},\\
b_{33} &= \frac{\partial u_3}{\partial \theta},\\
&\beta r'+T_1(2k_{18}\dot{b}_{11}+2k_{19}\dot{b}_{22}+2k_{20}\dot{b}_{33}+k_{21}\dot{T}')-\eta\frac{\partial^2 T'}{\partial \theta^2} = 0.
\end{aligned} \tag{6.26}
$$

Again, we note that these equations are of the same form as the corresponding equations of Green *et al.* [2].

7. STABILITY

The treatment of stability adopted here is based upon Liapounov's second method as extended to continuous systems by Movchan [4], Knops and Wilkes [5] and Gilbert and Knops [6]. The concept of stability is dynamic and roughly envisages that the system under examination is given an arbitrarily small perturbation at some definite instant. The magnitude, in an appropriate sense, of the subsequent disturbance, due to the initial perturbation, is used in the classification of stability or instability. Precise meanings are given to the magnitudes of the initial and subsequent disturbances by employing positive-definite functions ρ_τ, ρ respectively. In general, different functions are used for ρ_τ and ρ.

In order to give a definition of stability further ingredients are required. A time interval $\mathscr{T}_\tau$ must be prescribed at an instant of which the system is allowed to be initially disturbed. A second time interval $\mathscr{T}$ must also be prescribed during which the subsequent motion is examined for stability. In the situations considered here, both $\mathscr{T}_\tau$ and $\mathscr{T}$ are the intervals $[0, \infty)$, and the same results are obtained irrespective of when the system is initially disturbed. It is important to note that the initial disturbance is given to the system at one, and only one instant chosen from $\mathscr{T}_\tau$, whereas the subsequent motion is examined at all instants of $\mathscr{T}$.

Let the solution of the differential equations considered be denoted by the function ϕ. The function ϕ is defined on $\mathscr{T}$ and its values belong to some set X. Also let the initial value satisfied by a solution ϕ be denoted by $\phi(\tau)$, where $\phi(\tau)$ belongs to some set X_τ and $\tau \in \mathscr{T}_\tau$. Then at a time t later, the value of the solution is $\phi(\tau+t)$, $t \in \mathscr{T}$, $\tau \in \mathscr{T}_\tau$. We assume that $\phi(\tau+t)$ is in X, and that

$$\lim_{t\to 0} \phi(\tau+t) = \phi(\tau).$$

We write

$$\phi_\tau(t) = \phi(\tau+t) \qquad t \in \mathscr{T},$$

and observe that the function ϕ_τ is merely the translate of ϕ.

In this paper we are only concerned with the stability of the null solution of the system. Accordingly we only give a definition of stability of the null solution which is as follows: The null solution is stable, if for each $\tau \in \mathscr{T}_\tau$ and $\varepsilon > 0$, there exists $\delta(\varepsilon, \tau) > 0$ such that

$$\rho_\tau(\phi(\tau)) < \delta, \qquad \phi(\tau) \in X_\tau,$$

implies

$$\sup_{t\in\mathscr{T}} \rho(\phi_\tau(t)) < \varepsilon, \qquad t \in \mathscr{T}, \phi_\tau(t) \in X.$$

This definition is due to Movchan [4]. We note that the above definition has been generalised by Gilbert and Knops [6] and that the latter authors have related the whole concept of stability to that of continuity. When $\delta(\varepsilon, \tau)$ is independent of τ the stability is said to be uniform. It is evident from the definition that stability depends critically upon the explicit measures taken for ρ_τ and ρ. Indeed, it is possible for the null solution to be stable with respect to one pair of measures ρ_τ, ρ, and unstable with respect to a different pair.

It is often convenient to establish necessary and sufficient conditions for stability by means of a theorem analogous to the classical theorem of Liapounov. This theorem requires the introduction of a third set of positive-definite functions $F_{\tau,t}$ and is as follows: The null solution is stable if and only if for each $t \in \mathscr{T}$, $\tau \in \mathscr{T}_\tau$ there exists a positive definite function $F_{\tau,t}$ defined on X for which

$$\left.\begin{array}{l} \text{(i) given } \varepsilon > 0, \text{ there exists } \delta(\varepsilon,\tau) > 0 \text{ such that} \\ \qquad \rho_\tau(\phi(\tau)) < \delta \quad \text{implies} \quad F_{\tau,0}(\phi(\tau)) < \varepsilon, \\ \text{(ii) } F_{\tau,t}(\phi_\tau(t)) \text{ is non-increasing with respect to } t, \\ \text{(iii) given } \eta > 0, \text{ there exists } \xi(\eta,\tau) > 0 \text{ such that} \\ \qquad \sup_{t\in\mathscr{T}} F_{\tau,t}(\phi_\tau(t)) < \xi \quad \text{implies} \sup_{t\in\mathscr{T}} \rho(\phi_\tau(t)) < \eta. \end{array}\right\} \tag{7.1}$$

The above theorem has the corollary that the null solution is uniformly stable if and only if the conditions (i) and (ii) hold uniformly in τ.

A proof of this theorem and corollary have been given by Movchan [4] and Gilbert and Knops [6]. The latter work proves a more general result than that given above and imposes less restrictions than are required by Movchan [4].

We observe that when the system is stable two immediate candidates for $F_{\tau,t}$ are

$$\sup_{s\geq t} \rho(\phi_\tau(s)), \qquad \sup_{s\geq t} F_{\tau,s}(\phi_\tau(s)), \qquad s, t \in \mathscr{T}. \tag{7.2}$$

Hence condition (ii) enforces boundedness assumptions such as

$$\rho(\phi_\tau(t)) < \infty, \qquad F_{\tau,t}(\phi_\tau(t)) < \infty. \tag{7.3}$$

For practical purposes, it is often better to express condition (i) in the equivalent form

$$F_{\tau,0}(\phi(\tau)) \leq c_1 \rho_\tau(\phi(\tau)). \tag{7.4}$$

where c_1 is a positive constant, and condition (ii) in the equivalent form

$$F_{\tau,t}(\phi_\tau(t)) \geq c_2 \rho(\phi_\tau(t)), \tag{7.5}$$

where c_2 is a positive constant.

We now turn our attention to an explicit problem. We consider the stability of the initially straight rod which has been subjected to a simple extension and so we may use the results of section 6. As we have already remarked, the equations governing the small deformations of the rod separate into four distinct groups. In the remainder of this paper we shall only consider one of the groups of equations (6.22) which govern the (small) flexural vibrations in a plane normal to $\mathbf{A}_1$. Thus we shall permit only those initial perturbations which give rise to such flexural vibrations. However, the method indicated here can be readily applied to the remaining groups of equations and yield a complete stability analysis. We study the flexure problem because of the fair amount of interest in this problem in the framework of the classical Bernoulli–Euler rod theory.

If the body forces and director body forces are zero, we have, from (6.22), the two equations which govern the flexural vibrations:

$$(k_5 + A^{22}N^3)\frac{\partial^2 u}{\partial \theta^2} + k_5 \frac{\partial b}{\partial \theta} = \beta A^{22} \frac{\partial^2 u}{\partial t^2}, \tag{7.6}$$

$$k_{15}\frac{\partial^2 b}{\partial \theta^2} - k_5 b - k_5 \frac{\partial u}{\partial \theta} = \beta \alpha_2 A^{33} \frac{\partial^2 b}{\partial t^2}, \tag{7.7}$$

where, for simplicity, we have put

$$u \equiv u_2, \qquad b \equiv b_{23}. \tag{7.8}$$

We suppose that the rod is determined by $0 \leq \theta \leq l$ and put

$$\left.\begin{aligned} \theta &= lx, & u &= lv, \\ \frac{k_{15}}{k_5} &= \xi l^2, & \frac{A^{22}N^3}{k_{15}} &= -\frac{\lambda^2}{l^2}, \\ \frac{\beta A^{22}}{k_5} &= \frac{m}{l^2}, & \frac{\beta \alpha_2 A^{33}}{k_5} &= n. \end{aligned}\right\} \tag{7.9}$$

For the rest of this paper we assume that†

$$\xi > 0, \qquad m > 0, \qquad n > 0 \tag{7.10}$$

† We note that N^3 is positive for tension, negative for compression.

and that λ is real. Equations (7.6) and (7.7) now assume the more compact form

$$(1-\xi\lambda^2)\frac{\partial^2 v}{\partial x^2}+\frac{\partial b}{\partial x} = m\frac{\partial^2 v}{\partial t^2}, \tag{7.11}$$

$$\xi\frac{\partial^2 b}{\partial x^2}-b-\frac{\partial v}{\partial x} = n\frac{\partial^2 b}{\partial t^2}. \tag{7.12}$$

We examine the stability of the null solution $v \equiv 0$, $b \equiv 0$ when the rod is subject to one set of the boundary conditions

$$\text{(a)} \qquad v(0,t) = b(0,t) = 0, \qquad v(1,t) = b(1,t) = 0, \tag{7.13a}$$

$$\text{(b)} \qquad v(0,t) = \frac{\partial b}{\partial x}(0,t) = 0, \qquad v(1,t) = \frac{\partial b}{\partial x}(1,t) = 0. \tag{7.13b}$$

The boundary conditions (7.13a) are appropriate to a rod which is clamped at both ends, and (7.13b) are relevant to a rod which is simply supported at both ends. We notice that as a consequence of the differential equations (7.11), (7.12) and either set of boundary conditions (7.13), the total energy functional

$$E(t) = \int_0^1\left\{\xi\left(\frac{\partial b}{\partial x}\right)^2+b^2+(1-\xi\lambda^2)\left(\frac{\partial v}{\partial x}\right)^2+2b\frac{\partial v}{\partial x}+m\left(\frac{\partial v}{\partial t}\right)^2+n\left(\frac{\partial b}{\partial t}\right)^2\right\}\mathrm{d}x \tag{7.14}$$

does not vary in time, so that

$$E(t) = E(0).$$

In this application we take X to be the set of all real-valued functions, defined on the interval [0, 1] which are continuous together with their first and second derivatives. We let $\mathscr{T}_\tau = \mathscr{T} = [0, \infty)$ and arbitrarily take $\tau = 0$. It is readily seen that the subsequent conclusions are valid irrespective of the value of τ, so there is no loss of generality in assuming $\tau = 0$. Also we take the measure of the initial perturbation, ρ_τ, to be

$$\rho_\tau(\phi(\tau)) = E(0)^{\frac{1}{2}}.$$

We shall consider two choices of the subsequent disturbance, but in each case we shall use the functional† $E(t)^{\frac{1}{2}}$ to establish stability. The conditions of the stability theorem then demand that $E(t) < \infty$. Since the constant functional $E(t)^{\frac{1}{2}}$ is equal to the initial measure $\rho_\tau(\phi(\tau))$, the only condition of the stability theorem still to be satisfied is (7.5).

To carry out a detailed investigation we need some inequalities. First we consider the minimum of the functional

$$\int_0^1\left\{\left(f+\frac{\partial g}{\partial x}\right)^2+\xi\left(\frac{\partial f}{\partial x}\right)^2\right\}\mathrm{d}x \Big/ \int_0^1\left(\frac{\partial g}{\partial x}\right)^2\mathrm{d}x, \tag{7.15}$$

for functions $f, g \in X$ satisfying (7.13a) or (7.13b), i.e.

$$f(0,t) = f(1,t) = g(0,t) = g(1,t) = 0,$$

or

$$\frac{\partial f}{\partial x}(0,t) = \frac{\partial f}{\partial x}(1,t) = g(0,t) = g(1,t) = 0.$$

† That is, we shall put $F_{\tau,t}(\phi_\tau(t)) = E(t)^{\frac{1}{2}}$.

If the minimum is denoted by ξk^2, then for either set of boundary conditions we may show that the minimizing functions f_0, g_0 satisfy

$$\left.\begin{aligned} \xi\frac{\partial^2 f_0}{\partial x^2} - f_0 - \frac{\partial g_0}{\partial x} &= 0, \\ (1-\xi k^2)\frac{\partial^2 g_0}{\partial x^2} + \frac{\partial f_0}{\partial x} &= 0. \end{aligned}\right\} \tag{7.16}$$

We note that these equations are the static counterparts of (7.11) and (7.12). From (7.16) it is straightforward to show for clamped ends, (7.13a), that k is the smallest positive root of

$$k^2 = \frac{h^2}{1+\xi h^2}, \qquad \tan\frac{h}{2} = \frac{h}{2}\Big/(1+\xi h^2), \tag{7.17a}$$

whereas for simply supported ends, (7.13b), that

$$k^2 = \frac{\pi^2}{1+\xi\pi^2}. \tag{7.17b}$$

Thus for functions $b, v \in X$ we have

$$\int_0^1 \left\{\left(b+\frac{\partial v}{\partial x}\right)^2 + \xi\left(\frac{\partial b}{\partial x}\right)^2\right\} \mathrm{d}x \geq \xi k^2 \int_0^1 \left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x, \tag{7.18}$$

where k is given by (7.17a) for clamped ends and by (7.17b) for simply supported ends. Also we need the result† that for any function $f \in X$ such that $f(x, t) = 0$ for some $x \in [0, 1]$,

$$\int_0^1 \left(\frac{\partial f}{\partial x}\right)^2 \mathrm{d}x \geq \sup_{0\leq x\leq 1} |f(x, t)|^2. \tag{7.19}$$

Consider now the case (7.13a) when the rod has clamped ends. From (7.10), (7.14) and (7.18) we have

$$E(t) \geq \int_0^1 \left\{\xi\left(\frac{\partial b}{\partial x}\right)^2 + \left(b+\frac{\partial v}{\partial x}\right)^2 - \xi\lambda^2\left(\frac{\partial v}{\partial x}\right)^2\right\} \mathrm{d}x \tag{7.20}$$

$$\geq \int_0^1 \xi(k^2-\lambda^2)\left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}x, \tag{7.21}$$

where k is given by (7.17a). Thus provided

$$k^2 > \lambda^2 \tag{7.22}$$

we obtain from (7.13a), (7.19) and (7.21)

$$E(t)^{\frac{1}{2}} \geq [\xi(k^2-\lambda^2)]^{\frac{1}{2}} \sup_{0\leq x\leq 1} |v(x, t)|. \tag{7.23}$$

We have therefore established stability (in fact, uniform stability) with respect to the measures $E(0)^{\frac{1}{2}}$ and $\sup|v|$ provided (7.10) and (7.22) hold. Again, we have from (7.10),

† For a proof see for example Knops and Wilkes [5].

(7.13a), (7.14), (7.18) and (7.22)

$$\begin{aligned} E(t) &\geq \int_0^1 \frac{k^2-\lambda^2}{k^2}\left\{\left(b+\frac{\partial v}{\partial x}\right)^2+\xi\left(\frac{\partial b}{\partial x}\right)^2\right\}\mathrm{d}x \\ &\geq \int_0^1 \xi\frac{k^2-\lambda^2}{k^2}\left(\frac{\partial b}{\partial x}\right)^2\mathrm{d}x \end{aligned} \tag{7.24}$$

and with the help of (7.19) we get

$$E(t)^{\frac{1}{2}} \geq \left[\frac{\xi(k^2-\lambda^2)}{k^2}\right]^{\frac{1}{2}} \sup_{0\leq x\leq 1} |b(x,t)|. \tag{7.25}$$

Thus we have established uniform stability with respect to $E(0)^{\frac{1}{2}}$ and $\sup|b|$ under the same set of restrictions (7.10) and (7.21). We remark that we have shown that the rod is stable as long as (7.10) and (7.21) hold. This does not imply that if one or other of these conditions is violated then the rod is unstable.

Let us now discuss the case when the rod is simply supported so that boundary conditions (7.13b) hold. It is clear that we can repeat the calculation which starts at (7.20) and ends at (7.23) to establish uniform stability with respect to $\sup|v|$. The only change is that in this case k is given by (7.17b), and the stability condition becomes

$$\pi^2(1-\xi\lambda^2) > \lambda^2. \tag{7.26}$$

However, when we try to repeat the steps (7.24) to (7.25) we find that we are unable to do so, for in this case we cannot use (7.19) since b does not necessarily vanish for any $x \in [0, 1]$. Instead we proceed as follows: From (7.24)

$$\begin{aligned} E(t) &\geq \int_0^1 \xi\frac{k^2-\lambda^2}{k^2}\left(\frac{\partial b}{\partial x}\right)^2\mathrm{d}x \\ &= \int_0^1 \xi\frac{k^2-\lambda^2}{k^2}\left\{\frac{\partial}{\partial x}\left(b-\int_0^1 b\,\mathrm{d}x\right)\right\}^2\mathrm{d}x. \end{aligned}$$

Since, $b-\int_0^1 b\,\mathrm{d}x$ vanishes at least once for $x \in [0, 1]$ we may use (7.19) to obtain

$$E(t)^{\frac{1}{2}} \geq \left[\frac{\xi(k^2-\lambda^2)}{k^2}\right]^{\frac{1}{2}} \sup_{0\leq x\leq 1}\left|b(x,t)-\int_0^1 b\,\mathrm{d}x\right|.$$

But $\int_0^1 b\,\mathrm{d}x$ is independent of x and so

$$\begin{aligned} \sup_{0\leq x\leq 1}|b(x,t)| &\leq \left\{\frac{k^2}{\xi(k^2-\lambda^2)}E(t)\right\}^{\frac{1}{2}}+\left|\int_0^1 b\,\mathrm{d}x\right| \\ &\leq \left\{\frac{k^2}{\xi(k^2-\lambda^2)}E(t)\right\}^{\frac{1}{2}}+\left\{\int_0^1 b^2\,\mathrm{d}x\right\}^{\frac{1}{2}}, \end{aligned} \tag{7.27}$$

where we have used Schwartz's inequality. Inspection of (7.27) shows that if we can prove that $E(t)$ is bounded below by $\int_0^1 b^2\,\mathrm{d}x$ then we may establish stability with respect to $\sup|b|$. Of course, we would then incidentally have proved stability with respect to $\int_0^1 b^2\,\mathrm{d}x$.

To demonstrate the required result we consider the minimum of the functional

$$\int_0^1 \left\{(1-\xi\lambda^2)\left(\frac{f}{1-\xi\lambda^2}+\frac{\partial g}{\partial x}\right)^2+\xi\left(\frac{\partial f}{\partial x}\right)^2\right\} \mathrm{d}x \Big/ \int_0^1 f^2 \,\mathrm{d}x,$$

for functions $f, g \in X$ satisfying

$$\frac{\partial f}{\partial x}(0, t) = \frac{\partial f}{\partial x}(1, t) = 0, \qquad g(0, t) = g(1, t) = 0.$$

Note that (7.26) implies that $(1-\xi\lambda^2)$ is positive. Denoting the minimum by ξK^2, it may be shown in the usual way that the minimizing functions f_0, g_0 satisfy

$$\xi\frac{\partial^2 f_0}{\partial x^2}+\left(\xi K^2-\frac{1}{1-\xi\lambda^2}\right)f_0-\frac{\partial g_0}{\partial x} = 0,$$

$$\frac{1}{1-\xi\lambda^2}\frac{\partial f_0}{\partial x}+\frac{\partial^2 g_0}{\partial x^2} = 0,$$

and hence that

$$\xi K^2 = \min\left\{\xi\pi^2, \frac{1}{1-\xi\lambda^2}\right\}. \tag{7.28}$$

We therefore have the result that for functions $b, v \in X$, satisfying (7.13b),

$$\int_0^1 \left\{(1-\xi\lambda^2)\left(\frac{b}{1-\xi\lambda^2}+\frac{\partial v}{\partial x}\right)^2+\xi\left(\frac{\partial b}{\partial x}\right)^2\right\} \mathrm{d}x \geq \xi K^2 \int_0^1 b^2 \,\mathrm{d}x. \tag{7.29}$$

Now return to (7.14) to see that

$$E(t) \geq \int_0^1 \left\{\xi\left(\frac{\partial b}{\partial x}\right)^2+(1-\xi\lambda^2)\left(\frac{\partial v}{\partial x}+\frac{b}{1-\xi\lambda^2}\right)^2-\frac{\xi\lambda^2}{1-\xi\lambda^2}b^2\right\} \mathrm{d}x,$$

which, with the help of (7.28), gives

$$E(t) \geq \int_0^1 \xi\left(K^2-\frac{\lambda^2}{1-\xi\lambda^2}\right)b^2 \,\mathrm{d}x. \tag{7.30}$$

But (7.10) and (7.28) imply that

$$\xi\left(K^2-\frac{\lambda^2}{1-\xi\lambda^2}\right) > 0,$$

so from (7.27) and (7.30) we find that

$$\sup_{0\leq x\leq 1} |b(x, t)| \leq cE(t)^{\frac{1}{2}},$$

where c is a positive constant. Thus provided (7.10) and (7.26) are satisfied we may conclude uniform stability with respect to $E(0)^{\frac{1}{2}}$ and $\sup|b|$.

In the interpretation of the inequalities (7.10), (7.22) and (7.26) it must be remembered that k_5, k_{15} and N^3 are derived from the energy function, and hence depend upon A^{11}, A^{22}, A^{33}. In the special case, mentioned in section 6, when we regard the initial force to be obtained "without deformation" some explicit results may be stated since k_5 and k_{15}

are constants. For this case there is no loss of generality in taking $A^{11} = A^{22} = A^{33} = 1$, so we do. Also we make the reasonable assumptions† that

$$\beta > 0, \qquad \alpha_2 > 0, \qquad N^3 < 0; \tag{7.31}$$

then the inequalities (7.10) become

$$k_5 > 0, \qquad k_{15} > 0. \tag{7.32}$$

Granted (7.31) and (7.32), the stability condition for clamped ends is

$$-N^3 < k_{15}\frac{k^2}{l^2}, \tag{7.33}$$

where k is the smallest positive root of (7.17a), and the condition for simply supported ends is

$$-N^3 < \frac{k_5 k_{15}\pi^2}{k_5 l^2 + k_{15}\pi^2}. \tag{7.34}$$

We remarked in section 6 that the usual equations for the Euler strut could be obtained as a special case of our more general theory. Roughly, the classical theory is obtained by letting $k_5 \to \infty$ in the case when the initial force is obtained "without deformation". It is quite straightforward to obtain the critical loads in this case from (7.33) and (7.34) and it may be shown that the critical values obtained in the general development are smaller than the critical values corresponding to the classical theory.

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Абстракт—Дается теория для малых перемещений, накладываемых на большое перемещение упругого стерхня. Обсуждаются некоторые виды симметрии материала и геометрической симметрии. Получается решение некотоых задач в конечных перемещениях. Обсуждается такке устойчивость прямого стержня, подверженного простому большому удлиннению.

† Recall that $N^3 < 0$ implies that the rod is under compression.